

Bubble effect on Kelvin-Helmholtz' instability

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Abstract

We derive boundary conditions at interfaces (contact discontinuities) for a class of Lagrangian models describing, in particular, bubbly flows. We use these conditions to study Kelvin-Helmholtz' instability which develops in the flow of two superposed layers of a pure incompressible fluid and a fluid containing gas bubbles, co-flowing with different velocities. We show that the presence of bubbles in one layer stabilizes the flow in some intervals of wave lengths.

1 Introduction

Many mathematical models of fluid mechanics are derived through the approximation of the solution of boundary value problems for Euler equations. For example, equations for bubbly flows are derived as an approximation of the solution of a complex free-boundary problem describing the motion of the mixture of water and gas bubbles (Iordanski (1960), Kogarko (1961), Wijngaarden (1968)). Green-Naghdi's model describing wave motions of a liquid layer of finite depth in a shallow water approximation with account of dispersion effects is another example (Green *et al* (1974), Green & Naghdi (1976)). Due to the fact that averaging procedures and asymptotic expansions have been used in the derivation, it is not obvious to decide what boundary conditions are natural for these systems of equations. In this paper, a Hamiltonian formulation of the problem in (t, \mathbf{x}) -space is proposed which allows one to find boundary conditions for a general class of models. This class includes a model of bubbly fluid and dispersive shallow water. We use the boundary conditions to study the stability of two co-flowing layers of bubbly and pure fluids. In absence of gravity and capillarity, it is well known that Kelvin-Helmholtz' instability develops for any wave lengths in the flow of two superposed layers of pure incompressible fluids

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(see, for example, Drazin & Reid (1981)). We prove that the presence of bubbles in one layer can produce a stabilizing effect: the flow becomes stable with respect to perturbations of some wave lengths.

2 Governing equations and conditions on moving interfaces

2.1 Variation of Hamilton's action

Here we calculate the variation of Hamilton's action for a special class of Lagrangians. Usually this procedure is used only for derivation of governing equations. We obtain below not only governing equations but also boundary conditions at inner contact surfaces.

Let us consider the Lagrangian of the form:

$$L = L(\mathbf{J}, \frac{\partial \mathbf{J}}{\partial \mathbf{z}}, \mathbf{z}) \quad (2.1)$$

where $\mathbf{z} = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \equiv (z^i)$, $(i = 0, 1, 2, 3)$, t is the time, \mathbf{x} is the space variable, $\mathbf{J} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \end{pmatrix}$, ρ is the fluid density and \mathbf{u} is the velocity field. Let us calculate the variation of Hamilton's action in the case when the 4-D momentum \mathbf{J} verifies the equation of continuity:

$$Div \mathbf{J} \equiv \frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0 \quad (2.2)$$

Hamilton's action is defined by

$$a = \int_{\Omega} L d\mathbf{z} \quad (2.3)$$

where Ω is a material domain. Considering a smooth one-parameter family of virtual motions

$$\mathbf{z} = \Phi(\mathbf{Z}, \varepsilon), \quad \Phi(\mathbf{Z}, 0) = \varphi(\mathbf{Z})$$

(\mathbf{Z} stands for the Lagrangian coordinates, ε is a small parameter at the vicinity of zero and $\mathbf{z} = \varphi(\mathbf{Z})$ is the real motion), we define the virtual displacements $\zeta(\mathbf{Z})$ and the Lagrangian variations $\delta \mathbf{J}(\mathbf{Z})$ by the formulae:

$$\zeta(\mathbf{Z}) = \left. \frac{\partial \Phi(\mathbf{Z}, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}, \quad \delta \mathbf{J}(\mathbf{Z}) = \left. \frac{\partial \mathbf{J}(\mathbf{Z}, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \quad (2.4)$$

Due to the fact that $\mathbf{Z} = \varphi^{-1}(\mathbf{z})$, we consider the variations as functions of Eulerian coordinates and use the same notations $\zeta(\mathbf{z})$ and $\delta \mathbf{J}(\mathbf{z})$ in Eulerian variables. Hamilton's principle assumes that $\zeta(\mathbf{z}) = 0$ on the boundary $\partial\Omega$ of Ω .

In the following, the transposition is denoted by T . For any vectors \mathbf{a} and \mathbf{b} we use the notation $\mathbf{a}^T \mathbf{b}$ for their scalar product $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \mathbf{b}^T$ for their tensor product $\mathbf{a} \otimes \mathbf{b}$. The divergence of the second order tensor A is a covector defined by

$$Div(A\mathbf{h}) = Div(A)\mathbf{h}$$

where \mathbf{h} is any constant vector field. Div and div , $Grad$ and ∇ are respectively divergence and gradient operators in the 4- D and 3- D space. The identity matrix and the zero matrix of dimension n are denoted by I_n and O_n .

In calculations we shall use the equality

$$\delta \mathbf{J} = \left(\frac{\partial \zeta}{\partial \mathbf{z}} - (Div \zeta) I_4 \right) \mathbf{J} \quad (2.5)$$

which was proved in Gavriluk & Gouin (1999). Variation of the Hamilton action is

$$\delta a = \left. \frac{da}{d\varepsilon} \right|_{\varepsilon=0} = \int_{\Omega} \left(\delta L + L \operatorname{tr} \left(\frac{\partial \zeta}{\partial \mathbf{z}} \right) \right) d\mathbf{z} \quad (2.6)$$

Since

$$\delta L = \frac{\partial L}{\partial \mathbf{J}} \delta \mathbf{J} + \operatorname{tr} \left(\frac{\partial L}{\partial \left(\frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right)} \delta \left(\frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right) \right) + \frac{\partial L}{\partial \mathbf{z}} \zeta$$

and

$$\delta \left(\frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right) = \frac{\partial \delta \mathbf{J}}{\partial \mathbf{z}} - \frac{\partial \mathbf{J}}{\partial \mathbf{z}} \frac{\partial \zeta}{\partial \mathbf{z}}$$

we get from (2.5) and (2.6):

$$\delta a = \int_{\Omega} \left(\frac{\partial L}{\partial \mathbf{J}} \left(\frac{\partial \zeta}{\partial \mathbf{z}} - (Div \zeta) I_4 \right) \mathbf{J} + \operatorname{tr} \left(A^T \left(\frac{\partial \delta \mathbf{J}}{\partial \mathbf{z}} - \frac{\partial \mathbf{J}}{\partial \mathbf{z}} \frac{\partial \zeta}{\partial \mathbf{z}} \right) \right) + \frac{\partial L}{\partial \mathbf{z}} \zeta + L Div \zeta \right) d\mathbf{z}$$

with

$$A^T = \frac{\partial L}{\partial \left(\frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right)} \text{ or } (A^T)_i^j = \frac{\partial L}{\partial \left(\frac{\partial J^i}{\partial z^j} \right)} \quad (2.7)$$

For the sake of simplicity the measure of integration will not be indicated. Since for any linear transformation A and vector field \mathbf{v}

$$Div(A\mathbf{v}) = (Div A)\mathbf{v} + \operatorname{tr} \left(A \frac{\partial \mathbf{v}}{\partial \mathbf{z}} \right)$$

we get

$$\begin{aligned} \delta a &= \int_{\Omega} \operatorname{tr} \left(\left(\mathbf{J} \frac{\partial L}{\partial \mathbf{J}} - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right) \frac{\partial \zeta}{\partial \mathbf{z}} \right) + \left(L - \frac{\partial L}{\partial \mathbf{J}} \mathbf{J} \right) Div \zeta + \operatorname{tr} \left(A^T \frac{\partial \delta \mathbf{J}}{\partial \mathbf{z}} \right) + \frac{\partial L}{\partial \mathbf{z}} \zeta = \\ &= \int_{\Omega} \operatorname{tr} \left(\left(\mathbf{J} \frac{\partial L}{\partial \mathbf{J}} - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right) \frac{\partial \zeta}{\partial \mathbf{z}} \right) + \left(L - \frac{\partial L}{\partial \mathbf{J}} \mathbf{J} \right) Div \zeta + \end{aligned}$$

$$+Div (A^T \delta \mathbf{J}) - Div (A^T) \delta \mathbf{J} + \frac{\partial L}{\partial \mathbf{z}} \zeta ,$$

and finally

$$\begin{aligned} \delta a &= \int_{\Omega} tr \left(\left(\mathbf{J} \frac{\partial L}{\partial \mathbf{J}} - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right) \frac{\partial \zeta}{\partial \mathbf{z}} \right) + \left(L - \frac{\partial L}{\partial \mathbf{J}} \mathbf{J} \right) Div \zeta + Div (A^T \delta \mathbf{J}) - \\ &\quad - Div (A^T) \left(\frac{\partial \zeta}{\partial \mathbf{z}} - (Div \zeta) I_4 \right) \mathbf{J} + \frac{\partial L}{\partial \mathbf{z}} \zeta = \\ &= \int_{\Omega} tr \left(\left(\mathbf{J} \frac{\partial L}{\partial \mathbf{J}} - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} - \mathbf{J} Div (A^T) \right) \frac{\partial \zeta}{\partial \mathbf{z}} \right) + \\ &\quad + \left(L - \frac{\partial L}{\partial \mathbf{J}} \mathbf{J} + (Div A^T) \mathbf{J} \right) Div \zeta + Div (A^T \delta \mathbf{J}) + \frac{\partial L}{\partial \mathbf{z}} \zeta = \\ &= \int_{\Omega} Div \left(\left(\mathbf{J} \frac{\partial L}{\partial \mathbf{J}} - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} - \mathbf{J} Div (A^T) \right) \zeta + A^T \delta \mathbf{J} \right) - \\ &\quad - Div \left(\mathbf{J} \frac{\partial L}{\partial \mathbf{J}} - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} - \mathbf{J} Div (A^T) \right) \zeta + \\ &\quad + Div \left(\left(L - \frac{\partial L}{\partial \mathbf{J}} \mathbf{J} + (Div A^T) \mathbf{J} \right) \zeta \right) - Grad \left(L - \frac{\partial L}{\partial \mathbf{J}} \mathbf{J} + (Div A^T) \mathbf{J} \right)^T \zeta + \frac{\partial L}{\partial \mathbf{z}} \zeta \end{aligned}$$

Let us denote

$$\frac{\delta L}{\delta \mathbf{J}} \equiv \frac{\partial L}{\partial \mathbf{J}} - Div (A^T) = \mathbf{K}^T.$$

Then

$$\begin{aligned} \delta a &= \int_{\Omega} \left(\frac{\partial L}{\partial \mathbf{z}} - Div \left(\mathbf{J} \mathbf{K}^T - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} + (L - \mathbf{K}^T \mathbf{J}) I_4 \right) \right) \zeta + \quad (2.8) \\ &\quad + \int_{\partial \Omega} \mathbf{N}^T \left(\mathbf{J} \mathbf{K}^T - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} + (L - \mathbf{K}^T \mathbf{J}) I_4 \right) \zeta + \mathbf{N}^T A^T \delta \mathbf{J} \end{aligned}$$

with $\mathbf{N}^T = (-D_n, \mathbf{n}^T)$, D_n denotes the surface velocity and \mathbf{n} is the space unit normal vector. Virtual displacements vanish at the boundary $\partial \Omega$ and the surface integral is zero. The volume integral yields the equations of motion in conservative form as in Gavriluk & Gouin (1999).

In the case when fluid tensorial quantities are discontinuous at the inner interface Σ , expression (2.8) becomes

$$\begin{aligned} \delta a &= \int_{\Omega} \left(\frac{\partial L}{\partial \mathbf{z}} - Div \left(\mathbf{J} \mathbf{K}^T - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} + (L - \mathbf{K}^T \mathbf{J}) I_4 \right) \right) \zeta + \quad (2.9) \\ &\quad + \int_{\Sigma} \left[\mathbf{N}^T \left(\mathbf{J} \mathbf{K}^T - A^T \frac{\partial \mathbf{J}}{\partial \mathbf{z}} + (L - \mathbf{K}^T \mathbf{J}) I_4 \right) \zeta + \mathbf{N}^T A^T \delta \mathbf{J} \right] \end{aligned}$$

where the jump through Σ is denoted by $[\]$.

2.2 Governing equations and boundary conditions

We explicit the governing equations and the inner boundary conditions for the Lagrangian

$$L = \frac{1}{2}\rho|\mathbf{u}|^2 - W(\rho, \dot{\rho}), \quad \text{where } (\dot{}) = \frac{d}{dt} = \frac{\partial()}{\partial t} + \mathbf{u}^T \nabla() \quad (2.10)$$

Such a Lagrangian appears in the study of wave propagation in both shallow water flows with dispersion and bubbly flows (a complete discussion of these models is given in Gavriluk & Teshukov (2001)). We get

$$\frac{\partial \mathbf{J}}{\partial \mathbf{z}} = \begin{pmatrix} \frac{\partial \rho}{\partial t} & \frac{\partial \rho}{\partial \mathbf{x}} \\ \frac{\partial \mathbf{j}}{\partial t} & \frac{\partial \mathbf{j}}{\partial \mathbf{x}} \end{pmatrix}, \quad \frac{\partial L}{\partial \mathbf{J}} = \left(-\frac{|\mathbf{u}|^2}{2} - \frac{\partial W}{\partial \rho} + \frac{1}{\rho} \frac{\partial W}{\partial \dot{\rho}} (\nabla \rho)^T \mathbf{u}, \mathbf{u}^T - \frac{1}{\rho} \frac{\partial W}{\partial \dot{\rho}} (\nabla \rho)^T \right)$$

Matrix (2.7) becomes

$$A^T = \frac{\partial L}{\partial \left(\frac{\partial \mathbf{J}}{\partial \mathbf{z}} \right)} = \begin{pmatrix} -\frac{\partial W}{\partial \rho} & \mathbf{0}^T \\ -\frac{\partial W}{\partial \dot{\rho}} \mathbf{u} & O_3 \end{pmatrix}.$$

We get

$$\begin{aligned} Div(A^T) &= \left(-\frac{\partial}{\partial t} \left(\frac{\partial W}{\partial \rho} \right) - div \left(\frac{\partial W}{\partial \dot{\rho}} \mathbf{u} \right), \mathbf{0}^T \right), \\ \mathbf{K}^T &= \left(-\frac{|\mathbf{u}|^2}{2} - \frac{\delta W}{\delta \rho} + \frac{1}{\rho} \frac{\partial W}{\partial \dot{\rho}} (\nabla \rho)^T \mathbf{u}, \mathbf{u}^T - \frac{1}{\rho} \frac{\partial W}{\partial \dot{\rho}} (\nabla \rho)^T \right), \\ \mathbf{N}^T \mathbf{J} &= \rho (\mathbf{n}^T \mathbf{u} - D_n), \quad \mathbf{N}^T A^T = \left(-\frac{\partial W}{\partial \dot{\rho}} (\mathbf{n}^T \mathbf{u} - D_n), \mathbf{0}^T \right). \end{aligned} \quad (2.11)$$

We study the case when Σ is a contact surface and, consequently, $\mathbf{N}^T \mathbf{J} = 0$, $\mathbf{N}^T A^T = \mathbf{0}^T$. We see that the surface integral in (2.9) vanishes if the boundary conditions :

$$[L - \mathbf{K}^T \mathbf{J}] \equiv [p] = 0 \quad (2.12)$$

are fulfilled with $p \equiv L - \mathbf{K}^T \mathbf{J}$. Vanishing the volume integral in (2.9) and using relations (2.11), we obtain the governing equations in the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) &= 0, \\ \frac{\partial \rho \mathbf{u}}{\partial t} + div(\rho \mathbf{u} \mathbf{u}^T + p \mathbf{I}) &= 0, \end{aligned} \quad (2.13)$$

$$p \equiv \rho \left(\frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t} \left(\frac{\partial W}{\partial \dot{\rho}} \right) - \operatorname{div} \left(\frac{\partial W}{\partial \dot{\rho}} \mathbf{u} \right) \right) - W = \rho \frac{\delta W}{\delta \rho} - W.$$

A complete set of boundary conditions at the contact interfaces for the class of models considered is:

$$\mathbf{n}^T \mathbf{u} - D_n = 0, \quad [p] = 0 \quad (2.14)$$

Equations (2.13) have been obtained earlier in Gavriluk & Shugrin (1996) by means of the classical method of Lagrange multipliers. Notice that this method did not give jump conditions (2.12). System (2.13) is reminiscent of Euler equations for compressible fluids. The term p in equations of motion (2.13) and boundary conditions (2.14) stands for the pressure. Nevertheless, p is not a function of density as in the case of barotropic fluids: it depends also on material time derivatives of the density. Thus, p is not usual thermodynamic pressure. In general, it implies that the boundary conditions are not just consequences of conservation laws. For example, for Korteweg-type fluids where the stress tensor depends on the density gradient, the boundary conditions contain normal derivatives of the density at interfaces (Seppecher (1989), Gouin & Gavriluk (1999)). As we have shown here, the boundary conditions at inner interfaces in bubbly fluids do not depend on normal derivatives of the density: they involve only tangential derivatives.

If the flow domain is shared between two domains consisting of pure ideal fluid and bubbly fluid, all previous calculations are still valid and boundary conditions (2.14) can be extended to such a case.

2.3 Exact statement of the problem

Now, we propose to use conditions (2.14) to study Kelvin-Helmholtz' instability of a parallel flow between two rigid walls of a layer of bubbly fluid in contact with a layer of pure incompressible fluid.

Consider the 2-D flow of two superposed finite layers of inviscid fluids in the channel: $-\infty < x < +\infty$, $0 < y < H_0$. The first one is governed by the Euler equations :

$$\operatorname{div}(\mathbf{u}_1) = 0, \quad \frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1^T \nabla) \mathbf{u}_1 + \frac{\nabla p_1}{\rho_{10}} = 0, \quad -\infty < x < \infty, \quad 0 < y < h(t, x) \quad (2.15)$$

Here ρ_{10} is the constant density of pure fluid, $\mathbf{u}_1 = (u_1, v_1)^T$ is the velocity field and p_1 is the pressure.

In the domain $h(t, x) < y < H_0$ governing equations (2.13) for the second fluid are

$$\begin{aligned} \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{u}_2) &= 0, \\ \frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2^T \nabla) \mathbf{u}_2 + \frac{\nabla p_2}{\rho_2} &= 0, \\ p_2 &= \rho_2 \left(\frac{\partial W}{\partial \rho_2} - \frac{\partial}{\partial t} \left(\frac{\partial W}{\partial \dot{\rho}_2} \right) - \operatorname{div} \left(\frac{\partial W}{\partial \dot{\rho}_2} \mathbf{u}_2 \right) \right) - W, \quad W = W(\rho_2, \dot{\rho}_2) \end{aligned} \quad (2.16)$$

Here ρ_2 , $\mathbf{u}_2 = (u_2, v_2)^T$ and p_2 are the average density, velocity and mixture pressure, respectively; $W = W(\rho_2, \dot{\rho}_2)$ is a given potential. For the flow of compressible bubbles having the same radius, potential W has the form (see, for example, Gavriluk (1994) and Gavriluk & Teshukov (2001)):

$$W(\rho_2, \dot{\rho}_2) = \rho_2 \left(c_g \varepsilon_g(\rho_g) - 2\pi n \rho_l R^3 \dot{R}^2 \right) \quad (2.17)$$

where ε_g is the internal energy of the gas in bubbles, R is the bubble radius, ρ_g is the gas density, $\rho_l = \text{const}$ is the density of carrying phase, $c_g = \text{const}$ is the bubble mass concentration, n is the number of bubbles per unit mass. Potential W is the difference between the internal energy of gas and the kinetic energy of fluid due to radial bubble oscillations. The bubble radius and the bubble density are functions of the average density ρ_2 :

$$\frac{4}{3}\pi R^3 = \frac{1}{n} \left(\frac{1}{\rho_2} - \frac{1 - c_g}{\rho_l} \right), \quad \rho_g = c_g \left(\frac{1}{\rho_2} - \frac{1 - c_g}{\rho_l} \right)^{-1}$$

It can be shown that governing equations (2.16) with potential (2.17) coincide exactly with classical equations of bubbly fluids (Kogarko (1961) and van Wijngaarden (1968)). In particular, the role of "equation of state"

$$p_2 = \rho_2 \frac{\delta W}{\delta \rho_2} - W, \quad W = W(\rho_2, \dot{\rho}_2) \quad (2.18)$$

plays the Rayleigh-Lamb equation which governs the radial oscillations of a spherical bubble:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_l}(p_g - p_2), \quad p_g = \rho_g^2 \frac{d\varepsilon_g}{d\rho_g} \quad (2.19)$$

The equivalence between (2.19) and (2.18) with W given by (2.17) has been proved, for example, in Gavriluk (1994). Dissipation-free model (2.16)-(2.17) assumes that the sliding between components is negligible. Moreover, this model is valid only when the volume fraction of bubbles is very small. Notice that a detailed description of the bubble interaction has been recently done by Russo & Smereka (1996) and Herrero, Lucquin-Desreux & Perthame (1999) in the case of rigid bubbles, and Smereka (2002), Teshukov & Gavriluk (2002) in the case of compressible bubbles, by using a kinetic approach.

At the rigid walls $y = 0$ and $y = H_0$, the vertical components of the velocity are equal to zero

$$v_1(t, x, 0) = v_2(t, x, H_0) = 0.$$

In accordance with the previous Section, we prescribe the following boundary conditions at the contact interface $y = h(t, x)$:

$$h_t + u_1 h_x = v_1, \quad h_t + u_2 h_x = v_2, \quad p_1 = p_2 \quad (2.20)$$

3 Linear stability problem

3.1 Linearization

Consider the following main parallel flow of two fluids:

$$u_1 = u_{10} = \text{const}, \quad v_1 = 0, \quad p_1 = p_0 = \text{const}, \quad 0 < y < h_0, \quad h_0 = \text{const} \quad (3.1)$$

$$u_2 = u_{20} = \text{const}, \quad v_2 = 0, \quad p_2 = p_0 = \text{const}, \quad \rho_2 = \rho_{20} = \text{const}, \quad h_0 < y < H_0 \quad (3.2)$$

Small perturbations denoted by ' :

$$u_1 = u_{10} + u_1', \quad v_1 = v_1', \quad p_1 = p_0 + p_1',$$

$$\rho_2 = \rho_{20} + \rho_2', \quad u_2 = u_{20} + u_2', \quad v_2 = v_2', \quad p_2 = p_0 + p_2'$$

satisfy the linearized system of equations

$$u_{1x}' + v_{1y}' = 0, \quad \rho_{10} D_1 u_1' + p_{1x}' = 0, \quad \rho_{10} D_1 v_1' + p_{1y}' = 0, \quad 0 \leq y \leq h_0 \quad (3.3)$$

$$D_2 \rho_2' + \rho_{20} (u_{2x}' + v_{2y}') = 0, \quad \rho_{20} D_2 u_2' + p_{2x}' = 0, \quad \rho_{20} D_2 v_2' + p_{2y}' = 0, \quad h_0 \leq y \leq H_0. \quad (3.4)$$

We use the notations

$$D_i = \frac{\partial}{\partial t} + u_{0i} \frac{\partial}{\partial x}, \quad i = 1, 2$$

In (3.4)

$$p_2' = a^2 \rho_2' + b^2 D_2^2 \rho_2'$$

with

$$a^2 = \rho_{20} \frac{\partial^2 W}{\partial \rho_2^2}(\rho_{20}, 0), \quad b^2 = -\rho_{20} \frac{\partial^2 W}{\partial \rho_2}(\rho_{20}, 0).$$

Here a is the *equilibrium* sound velocity, and b is a characteristic wave length depending on the bubble radius and gas volume fraction. We suppose that

$$\frac{\partial^2 W}{\partial \rho_2 \partial \rho_2}(\rho_{20}, 0) = 0$$

This condition is obviously fulfilled for potential (2.17). The coefficients a and b calculated in equilibrium $R = R_0$ and $p_2 = p_0$ are :

$$a^2 = -\frac{dp_g}{d\tau}(\tau_0) \frac{1}{N_0 \rho_{20}}, \quad b^2 = \frac{1}{4\pi N_0 R_0} \frac{\rho_l}{\rho_{20}} \quad (3.5)$$

For dilute mixtures we can simplify these expressions by replacing ρ_{20} by ρ_l :

$$a^2 = -\frac{dp_g}{d\tau}(\tau_0) \frac{1}{N_0 \rho_l}, \quad b^2 = \frac{1}{4\pi N_0 R_0} = \frac{R_0^2}{3\alpha_0}$$

Here $N_0 = \rho_{20}n$ is the number of bubbles per unit volume, $R = R_0$ is the equilibrium bubble radius, $\alpha_0 = \tau_0 N_0$ is the volume fraction of gas, $\tau_0 = \frac{4}{3}\pi R_0^3$ is the bubble volume and $p_g = p_g(\tau)$ is the gas pressure in bubbles expressed as a function of bubble volume. The equilibrium sound speed a is usually small with respect to the gas sound velocity. Expressions (3.5) can be obtained directly after linearization of Rayleigh-Lamb's equation for bubbles (2.19).

Boundary conditions (2.20) at $y = h_0$ are

$$h'_t + u_{10}h'_x = v'_1, \quad h'_t + u_{20}h'_x = v'_2, \quad p'_1 = p'_2 \quad (3.6)$$

Let us consider the normal modes of the linear problem (3.3), (3.4), (3.6):

$$u'_1 = U_1(y) \exp(i(kx - \omega t)), \quad v'_1 = ikV_1(y) \exp(i(kx - \omega t)), \quad (3.7)$$

$$p'_1 = P_1(y) \exp(i(kx - \omega t)), \quad h' = H \exp(i(kx - \omega t))$$

$$\rho'_2 = R_2(y) \exp(i(kx - \omega t)), \quad u'_2 = U_2(y) \exp(i(kx - \omega t)),$$

$$v'_2 = ikV_2(y) \exp(i(kx - \omega t)), \quad p'_2 = P_2(y) \exp(i(kx - \omega t))$$

By substituting into the linearized system we get the system of equations for unknown amplitudes

$$U_1 + V_{1y} = 0, \quad \rho_{10}(u_{10} - c)U_1 + P_1 = 0, \quad (3.8)$$

$$-\rho_{10}k^2(u_{10} - c)V_1 + P_{1y} = 0,$$

$$(u_{20} - c)R_2 + \rho_{20}(U_2 + V_{2y}) = 0, \quad \rho_{20}(u_{20} - c)U_2 + P_2 = 0,$$

$$-\rho_{20}k^2(u_{20} - c)V_2 + P_{2y} = 0, \quad P_2 = (a^2 - b^2k^2(u_{20} - c)^2) R_2$$

with the following boundary conditions

$$V_1(0) = 0, \quad V_2(H_0) = 0, \quad P_1(H_0) = P_2(H_0), \quad \frac{V_1(h_0)}{u_{10} - c} = \frac{V_2(h_0)}{u_{20} - c} = H$$

Here $c = \frac{\omega}{k}$ is the phase velocity.

From equations (3.8) we obtain the eigenvalue problem for pressure amplitudes:

$$P_{1yy} - k^2 P_1 = 0, \quad 0 < y < h_0, \quad (3.9)$$

$$P_{2yy} - \lambda^2 k^2 P_2 = 0, \quad h_0 < y < H_0,$$

$$P_1(h_0) = P_2(h_0), \quad \frac{P_{1y}(h_0)}{\rho_{10}(u_{10} - c)^2} = \frac{P_{2y}(h_0)}{\rho_{20}(u_{20} - c)^2}, \quad P_{1y}(0) = P_{2y}(H_0) = 0$$

where

$$\lambda^2 = 1 - \frac{(u_{20} - c)^2}{a^2 - b^2k^2(u_{20} - c)^2}.$$

It follows from (3.9) that the eigenvalues c are solutions of the equation

$$\frac{th(kh_0)}{\rho_{10}(u_{10} - c)^2} + \frac{\lambda th(\lambda k(H_0 - h_0))}{\rho_{20}(u_{20} - c)^2} = 0 \quad (3.10)$$

Next we assume that both layers are thin: $k(H_0 - h_0) \ll 1$, $kh_0 \ll 1$, and dispersion relation (3.10) is simplified into

$$\frac{(H_0 - h_0)}{\rho_{20}} \left(\frac{1}{(u_{20} - c)^2} - \frac{1}{a^2 - b^2 k^2 (u_{20} - c)^2} \right) + \frac{h_0}{\rho_{10}(u_{10} - c)^2} = 0 \quad (3.11)$$

3.2 Study of dispersion relation

In case $u_{20} = u_{10}$, equation (3.11) reduces to the quadratic equation

$$(1 + A)a^2 - ((1 + A)b^2 k^2 + 1)(u_{20} - c)^2 = 0, \text{ with } A = \frac{h_0 \rho_{20}}{(H_0 - h_0) \rho_{10}}.$$

which has only real roots. It means that the flows with equal velocities are stable.

Let us consider the general case when $u_{20} \neq u_{10}$. Equation (3.11) can be rewritten as a polynomial of fourth degree. The stability needs that all four roots are real. To study this problem, it is convenient to rewrite the dispersion relation (3.11) in the form

$$F(z) + A \equiv 0 \quad (3.12)$$

with

$$F(z) = (1 - Nz)^2 \left(1 - \frac{d^2}{z^2 - 1} \right), \quad (3.13)$$

$$z = \frac{ad}{u_{20} - c}, \quad d = \frac{1}{bk}, \quad N = \frac{M}{d}, \quad M = \frac{u_{20} - u_{10}}{a}$$

Here M is similar to the Mach number, d is the dimensionless length of perturbation wave. Obviously, z is real if and only if c is real. The derivative of (3.13) is

$$F'(z) = -2Nd^2(1 - Nz)\varphi(z) \quad (3.14)$$

where

$$\varphi(z) = \frac{1}{d^2} + \frac{1 - N^{-1}z}{(z^2 - 1)^2}. \quad (3.15)$$

Four cases must be considered

$$1^\circ \quad 0 < N < 1, \quad d^2 \leq N^{-2} - 1 \quad (3.16)$$

$$2^\circ \quad 0 < N < 1, \quad d^2 > N^{-2} - 1 \quad (3.17)$$

$$3^\circ \quad N > 1, \quad d^2 > \frac{16N^2}{27} \left((8N^2 - 6) \sqrt{1 - \frac{3N^{-2}}{4}} + 8N^2 - 9 \right) \quad (3.18)$$

$$4^\circ \quad N > 1, \quad d^2 \leq \frac{16N^2}{27} \left((8N^2 - 6) \sqrt{1 - \frac{3N^{-2}}{4}} + 8N^2 - 9 \right) \quad (3.19)$$

Each case is illustrated in Figures 1-4. For greater convenience, the roots z^i of (3.12) are shown as the intersection points of the two graphs $\eta = F(z)$ and $\eta = -A$.

The results are the following:

In case 2 the dispersion relation defined by (3.12)-(3.13) has four real roots for A not large.

In case 3 the dispersion relation has four real roots for "intermediate" values of A .

In other cases the dispersion relation has two real roots and two complex roots.

Precise formulations are given in the following Propositions.

Proposition 1 *If inequalities (3.16) are satisfied, equation (3.12) has two real roots and two complex roots for any positive A (Figure 1).*

Proposition 2 *If inequalities (3.17) are satisfied, the equation $\varphi(z) = 0$ defined by (3.15) has one root z_{01} on interval I_2 . Equation (3.12) has four real roots for $0 < A \leq -F(z_{01})$ (Figure 2a) and two real and two complex roots for $A > -F(z_{01})$ (Figure 2b).*

Proposition 3 *If inequalities (3.18) are satisfied, the equation $\varphi(z) = 0$ defined by (3.15) has two roots z_{02} and z_{03} on interval I_2 . Equation (3.12) has four real roots for $-F(z_{02}) < A \leq -F(z_{03})$ (Figure 3b), two real and two complex roots for $A < -F(z_{02})$ (Figure 3a) and $A > -F(z_{03})$ (Figure 3c).*

Proposition 4 *If inequalities (3.19) are satisfied, equation (3.12) has two real roots and two complex roots for any positive A (Figure 4).*

The proofs are given in Appendix.

It is convenient to rewrite conditions (3.16)-(3.19) in terms of b , k and M :

$$1^\circ : 0 < M < 1, \quad k < \frac{\sqrt{1-M^2}}{bM} \quad (3.20)$$

$$2^\circ : 0 < M < 1, \quad \frac{\sqrt{1-M^2}}{bM} < k < \frac{1}{bM} \quad \text{or} \quad M > 1, \quad k < \frac{1}{bM} \quad (3.21)$$

$$3^\circ : \frac{1}{bM} < k < \frac{1}{bd_0(M)} \quad (3.22)$$

$$4^\circ : \frac{1}{bd_0(M)} < k \quad (3.23)$$

Here $d_0(M)$ is the root of the equation $f(d, M) = 0$, where

$$f(d, M) = d^2 - \frac{16M^2}{27d^2} \left(\left(8 \left(\frac{M}{d} \right)^2 - 6 \right) \sqrt{1 - \frac{3d^2}{4M^2}} + 8 \left(\frac{M}{d} \right)^2 - 9 \right)$$

($0 < d_0(M) < M$). This equation has a unique solution $d = d_0(M)$ in a domain $0 < d < M$ because of the properties $f_d(d, M) > 0$, $f(d, M) \rightarrow -\infty$ as $d \rightarrow 0$ and $f(M, M) = M^2 > 0$.

Inequalities (3.20)-(3.23) are illustrated in Figure 5. For any given flow parameters M and b , one can find a sufficiently large wave number k satisfying inequality (3.23). It means that the flow is always unstable with respect to perturbations with sufficiently short wave lengths (as it follows from Proposition 4, the dispersion relation has complex roots). The flow with subsonic relative velocity is stabilized in an intermediate interval of wave lengths if the pure liquid layer is thin enough with respect to the bubbly layer. In the case of supersonic relative velocity a similar stabilization is observed for long waves (see Proposition 2 and inequalities (3.21)). For another intermediate interval of wave lengths (3.22) the stabilization of perturbations is attained for layer depths of the same order (see Proposition 3).

4 Conclusion and discussion

We have derived from Hamilton's principle of stationary action governing equations and boundary conditions at the contact interfaces in bubbly fluids. It has been shown that the dynamic condition on the interface reduces to the continuity of the average pressure. By using the boundary conditions derived, we have studied the Kelvin-Helmholtz instability of two superposed layers of a pure incompressible fluid and a bubbly fluid. We have shown that in contrast to the case of two incompressible fluids when the instability develops for any length of perturbations (if the gravity, capillarity or compressibility are not taken into account), the presence of bubbles can stabilize the flow in some range of perturbation wave lengths.

The stabilizing effect is due to the following reason. The development of classical Kelvin-Helmholtz instability leads to the appearance of wave-like bulges at the interface between two fluids. The presence of bubbles in a fluid (i.e. new interfaces) permits one to transform a part of the energy responsible for the bulge formation into the energy of radial oscillations of bubbles.

The viscosity effect on the flow stabilization is an important issue. The method of viscous potential flows developed by Joseph *et al* (1999) in the analysis of Rayleigh-Taylor and applied by Funada & Joseph (2001) in the analysis of Kelvin-Helmholtz, could be used here.

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6 Appendix

Real roots of equation (3.12) belong to the intervals

$$I_1 : -(1 + d^2)^{1/2} < z < -1, \quad I_2 : 1 < z < (1 + d^2)^{1/2}$$

Assuming that $u_{20} > u_{10}$ and the wave number k is positive, we see that the relative Mach number M and the constant d are positive. Equation (3.12) has always at least two real roots for any positive values of d, A and N because on the real axis $F(z)$ tends to $-\infty$ as $|z|$ tends to $1 + 0$, and $F(z) = 0$ for $|z| = (1 + d^2)^{1/2}$. Additional two real roots can appear in subsets of real axis where $F'(z)$ changes sign.

Notice, that the interval I_1 contains exactly one root of (3.12) because, obviously, $F'(z) < 0$ for $z < -1$ (see (3.14)). We prove below that, depending on the flow parameters, equation (3.12) has one or three real roots on interval I_2 .

First, let us show that $\varphi'(z) > 0$ on I_2 if $0 < N < 1$. If $\sqrt{3}/2 < N < 1$, then the derivative

$$\varphi'(z) = \frac{3z^2 - 4Nz + 1}{N(z^2 - 1)^3}$$

vanishes at points $z_1 < z_2$, where

$$z_{2,1} = \frac{2N}{3} \left(1 \pm \sqrt{1 - \frac{3N^{-2}}{4}} \right). \quad (6.1)$$

The inequality $\varphi'(z) > 0$ is fulfilled because in this case the roots z_i satisfy the inequalities $0 < z_1 < z_2 < 1$, and the interval where $\varphi'(z) < 0$ has no common points with I_2 . For $0 < N < \sqrt{3}/2$ the roots z_i of the equation $\varphi'(z) = 0$ are complex and $\varphi'(z) > 0$ on I_2 .

Proof of Proposition 1. It follows from inequalities (3.16) that $1 - Nz > 0$ for $z \in I_2$. Let us show that $F'(z) > 0$ on this interval. It was proved above that $\varphi'(z) > 0$ on I_2 if $0 < N < 1$. Taking into account that

$$\varphi(\sqrt{d^2 + 1}) = \frac{\sqrt{d^2 + 1}}{Nd^4} (N\sqrt{d^2 + 1} - 1) < 0 \quad (6.2)$$

we obtain inequalities $\varphi(z) < 0$, $F'(z) > 0$ on interval $(1, \sqrt{d^2 + 1})$. Then the equation (3.12) has only one root on I_2 . Proposition 1 is proved.

Proof of Proposition 2. In this case $N^{-1} \in I_2$, $F'(z) > 0$ for $1 < z < N^{-1}$ and $F(z)$ has a local maximum at $z = N^{-1}$, where $F'(N^{-1}) = 0$, $F(N^{-1}) = 0$. Taking into account that $\varphi'(z) > 0$, $\varphi(\sqrt{d^2 + 1}) > 0$ and $\varphi(N^{-1}) = \frac{(d^2 + 1)N^2 - 1}{d^2(N^2 - 1)} < 0$, we see that there exists the unique point z_{01} on I_2 , $z_{01} \in (N^{-1}, \sqrt{d^2 + 1})$, such that $\varphi(z_{01}) = 0$, $F'(z_{01}) = 0$. At this point $F(z)$ takes a local negative minimum: $F(z_{01}) < 0$. It is obvious that equation (3.12) has three roots on I_2 only if $0 < A < -F(z_{01})$ (see Figure 2a). Proposition 2 is proved.

Proof of Proposition 3. When inequalities (3.18) are satisfied, the function $\varphi(z)$ is positive in a neighbourhood of the end points of I_2 ($\varphi(z) \rightarrow \infty$ as $z \rightarrow 1 + 0$, $\varphi(\sqrt{d^2 + 1}) > 0$) and $1 - Nz < 0$. The function $F(z)$ can be non monotone only in the case when $\varphi(z)$ takes negative values and, consequently, $\varphi'(z)$ vanishes at some point of I_2 . It follows from (6.1) that the roots of the equation $\varphi'(z) = 0$ are such that $z_1 < 1$, $z_2 > 1$ for $N > 1$. It is easy to verify that a stronger inequality $z_2 > N$ is valid for $N > 1$. Hence only the root z_2 of the equation $\varphi'(z) = 0$ can belong to I_2 . The inequalities

$$a) d^2 > z_2^2 - 1, \quad b) \varphi(z_2) = \frac{1}{d^2} + \frac{1 - N^{-1}z_2}{(z_2^2 - 1)^2} < 0 \quad (6.3)$$

provide the inclusion $z_2 \in I_2$ and the existence of the roots z_{02} , z_{03} of the equations $\varphi(z_{02}) = F'(z_{02}) = 0$, $\varphi(z_{03}) = F'(z_{03}) = 0$ satisfying inequalities $1 < z_{02} < z_{03} < \sqrt{d^2 + 1}$. The function $F(z)$ has a local maximum at z_{02} and a local minimum at z_{03} . It means that equation (3.12) has three roots on I_2 if and only if $-F(z_{03}) < A < -F(z_{02})$ (see Figure 3a). For $A > -F(z_{02})$ and $0 < A < -F(z_{03})$ this equation has only one root (see Figures 3b, 3c). Notice that inequality (6.3a) is a consequence of (6.3b) because for $N > 1$

$$\frac{(z_2^2 - 1)^2}{N^{-1}z_2 - 1} > z_2^2 - 1 \quad (6.4)$$

Using the identity

$$\frac{(z_2^2 - 1)^2}{N^{-1}z_2 - 1} = \frac{16N^2}{27} \left((8N^2 - 6) \sqrt{1 - \frac{3N^{-2}}{4}} + 8N^2 - 9 \right)$$

one can show that inequality (6.3b) is equivalent to the second inequality in (3.18). Proposition 3 is proved.

Proof of Proposition 4. In accordance with (6.4) two cases are possible:

$$a) \frac{(z_2^2 - 1)^2}{N^{-1}z_2 - 1} > d^2 > z_2^2 - 1, \quad b) \frac{(z_2^2 - 1)^2}{N^{-1}z_2 - 1} > z_2^2 - 1 > d^2 \quad (6.5)$$

Inequalities (6.5a) are equivalent to the inclusion $z_2 \in I_2$ (see (6.3)) and the positiveness of minimal value of $\varphi(z)$ on I_2 . Using the inequalities $\varphi(z) > 0$, $1 - Nz < 0$ and (3.14) we show that $F'(z) > 0$. Consequently, equation (3.12) has only one root on I_2 .

If inequality (6.5b) is fulfilled, then $z_2 \notin I_2$ and $\varphi'(z)$ does not change sign on I_2 . The function $\varphi(z)$ is positive on I_2 , and $F(z)$ is a monotone function. We see that equation (3.12) also has only one root on I_2 . Proposition 4 is proved.

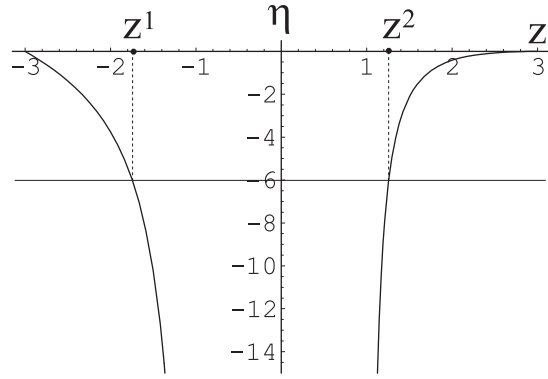


Figure 1: The variation of F as a function of z for case 1. The dispersion relation $F(z) + A = 0$ has only two real roots z^i , $i = 1, 2$ for any positive A . Graph is drawn for $N = 0.25$ and $d = 2\sqrt{2}$.

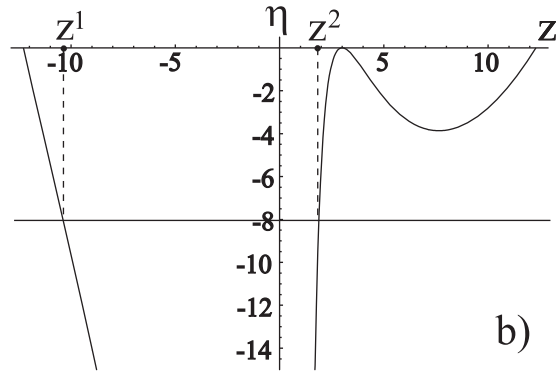
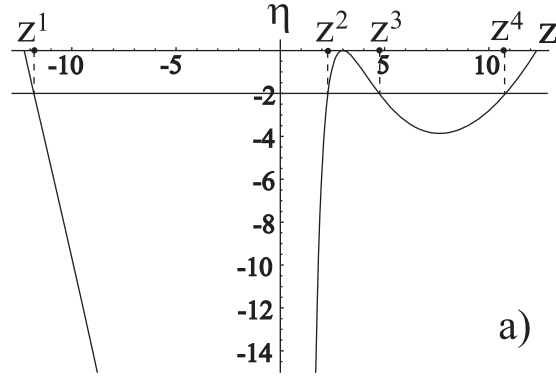


Figure 2: The variation of F as a function of z for case 2. The dispersion relation $F(z) + A = 0$ has four real roots $z^i, i = 1, 2, 3, 4$ for not large values of A and two real roots $z^i, i = 1, 2$ for large values of A . Graph is drawn for $N = 3$ and $d = \sqrt{150}$.

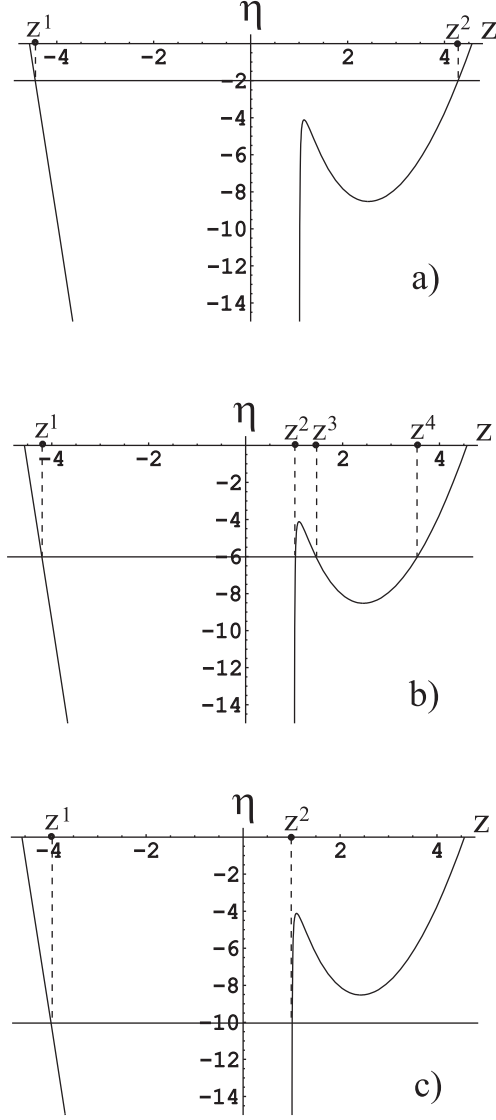


Figure 3: The variation of F as a function of z for case 3. The dispersion relation $F(z) + A = 0$ has two real roots z^i , $i = 1, 2$ for A small or large, and four real roots z^i , $i = 1, 2, 3, 4$ for A intermediate. Graph is drawn for $N = 1.1$ and $d = 4.45$.

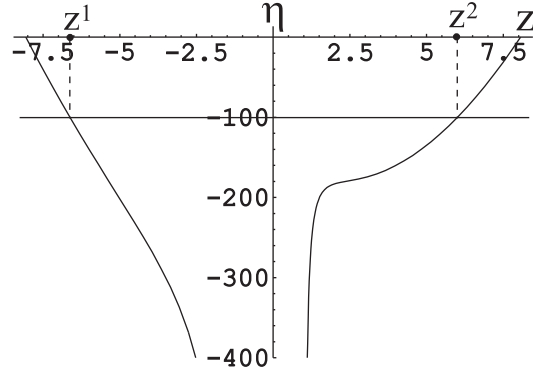


Figure 4: The variation of F as a function of z for case 4. The dispersion relation $F(z) + A = 0$ has only two real roots z^i , $i = 1, 2$ for any positive A . Graph is drawn for $N = 2$ and $d = 10.5$

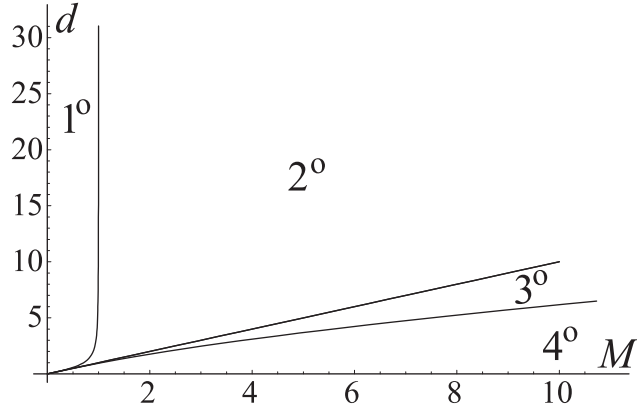


Figure 5: Cases 1-4 determined by inequalities (3.20)-(3.23) define four connected sets in (M, d) -plane. In regions 2^- and 3^- four real roots can exist.